J. DIFFERENTIAL GEOMETRY
26 (1987) 33–66

INSTABILITY OF THE LIOUVILLE PROPERTY FOR QUASI-ISOMETRIC RIEMANNIAN MANIFOLDS AND REVERSIBLE MARKOV CHAINS

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0. Introduction

It is shown by example that the Liouville property is not a quasi-isometry invariant of Riemann manifolds or for reversible Markov chains. Thus the example illustrates the subtleties involved in trying to understand the global function theory of Riemannian manifolds in terms of the behavior of discrete combinatorial models.

Let M be a manifold, ρ a complete Riemannian metric, and Δ_ρ the associated Laplacian operator. Many global function theoretic properties of Δ_ρ have geometric significance. This paper is concerned with the changes in the function theory which occurs as ρ is replaced by a quasi-isometrically equivalent metric τ; that is there exists a $C > 1$ such that for all $u \in TM_x$, for all x in M, we have $1/C < \rho(u, u)/\tau(u, u) < C$.

In parallel with manifolds we consider reversible Markov chains. These are defined by specifying a positive symmetric rate function $(a_{ij})_{i,j \in X}$ on a countable set X so that $\pi_i = \sum_{j \in X} a_{ij} < \infty$. These then admit the finite

Received July 9, 1985 and, in revised form, April 14, 1986.

difference analogue of a Laplace operator:

$$(\Delta_a f)_i = \sum_j a_{ij}(f_j - f_i),$$

acting on functions f from X into $\mathbb{R}$.

Again one can ask how the function theory of Δ_a compares with Δ_b where there is a $C > 1$ such that $C^{-1} < a_{ij}/b_{ij} < C$ for all $i, j \in X$ (where $0/0 = 1$ by convention).

There are comparisons in both directions between manifolds and reversible chains (see Royden, Varopolous, and Lyons & Sullivan [11], [14], [9] for examples where a manifold problem is reduced to a discrete situation). §4 of this paper goes in the other direction and gives one recipe for constructing manifolds which "look like" reversible Markov chains (see H. Watanabe [15] for another). However all these constructions fail to distinguish between quasi-equivalent metrics on M or quasi-equivalent rate functions on X. It seems important therefore, that one should understand how the function theoretic properties of Δ are affected by these perturbations.

The properties which concern us in this paper are the Liouville properties. We say M or X has the strong (weak) Liouville property if X has no nonconstant positive (bounded) solutions h to $\Delta h = 0$ (henceforth such solutions will be called harmonic even in the discrete case). We shall prove by example that it can happen that (M, ρ) has the strong Liouville property and (M, τ) fails to have the weak Liouville property while ρ is quasi-equivalent to τ. This completely settles a problem posed by Royden [11], [12]. A partial solution involving nonhomeomorphic Martin Boundaries was given by [13] using ideas of [1] in a fundamental way.

Not all properties are badly behaved, and under restricted conditions the Liouville property is also well behaved. It is known [9], [8] that the existence of a Green function is a quasi-isometry invariant for complete manifolds and for reversible Markov chains, and understandably it is in this context that the papers [12], [14] made use of discrete approximations to the manifolds. For planar Riemann surfaces either of the Liouville properties is equivalent to having a Green function and so each is a quasi-isometry invariant. Moser's Harnack theorem [10] proves that any surface quasi-isometric to $\mathbb{R}^d$ with its usual metric possesses the strong Liouville property and Kanai [6] has extended this to manifolds roughly isometric to $\mathbb{R}^d$. All the examples are consistent with the following positive

Conjecture. *The Liouville property is a quasi-isometry invariant among manifolds of polynomial volume growth.*

Certainly the example here has exponential growth. If such a conjecture were true it would widen the class of surfaces having the Liouville property quite considerably—for many special cases are known (e.g. Yau [16] has proved that any manifold of nonnegative curvature has the weak Liouville property).

There is a well-known dictionary relating the potential theoretic properties (Liouville, Green Function, etc.) and associated probabilistic statements (trivial shift invariant tail σ-field, transience, etc.) and in the case of constant negative curvature with ergodic properties of the geodesic flow. We give the following easy additional characterization of the weak Liouville property: If in M there is a surface S with $M \setminus S$ having two components M_1, M_2 and two points $x_i \in M_i$ such that the probability of hitting S from x_i is strictly less than one for $i = 1, 2$, then M does not have the weak Liouville property. If no such surface exists, then M does have the weak Liouville property. (On the one hand let S be a level set for the harmonic function; on the other let $h(x)$ be the probability starting from x that Brownian motion finishes on the component M_1.)

We give one final motivation for studying quasi-equivalence of Markov chains. Let Λ be a finitely generated group and $(g_i)_{i=1}^n$ a set of generators. If ν is a measure on Λ supported by $(g_i^{\pm 1})_{i=1}^n$, then ν defines a Markov chain by

$$\mathbb{P}(X_n = gX_{n-1}) = \nu(g).$$

Suppose ν is symmetric (that is to say $\nu(g) = \nu(g^{-1})$ for each $g \in \Lambda$) and that the minimal support of ν is $(g_i^{\pm 1})_{i=1}^n$. Then the walk is reversible. What properties of the random walk are independent of the choice of symmetric ν (or of (g_i))—in other words what properties of the walk are algebraic invariants of Λ? Recurrence is one such property. We say $A \subset \Lambda$ is ω-absorbing if X_n eventually enters A and stays there for ever (that is $\complement A$ is thin at infinity). For the free group on two generators the property of being ω-absorbing is not independent of ν! It would be most interesting to know whether the property of being ω-absorbing is an algebraic invariant for abelian, nilpotent, soluble, or amenable groups. (Note: one only considers random walks coming from symmetric ν.) If it failed to be invariant even for nilpotent groups, then much simpler examples for instability of the Liouville properties than the one given in this paper would be available.

We now summarize our main example and outline the structure of the paper. The main theme will be to construct Markov chain examples and then build manifolds which look like the Markov chains.

§1 introduces reversible Markov chains more fully than here.

§2 introduces a simple pair of quasi-equivalent reversible Markov chains exhibiting instability of the weak Liouville property. Although this example is unsatisfactory in many ways and will not carry over to manifolds, we learn one important point. Suppose X is a countable set admitting two quasi-equivalent reversible Markov structures $\mathbf{a}$, $\mathbf{b}$. Suppose $(X, \mathbf{a})$ and $(X, \mathbf{b})$ are both Liouville. If there is a subset $A \subset X$ which is ω-absorbing for $(X, \mathbf{a})$ while not for $(X, \mathbf{b})$, then there is a simple construction of a new pair of Markov chains exhibiting the required instability of the Liouville property.

§3 considers simple random walks on the free group Γ presented by the two generators a and b without relations. If the walk is generated by a symmetric measure ν on $\{a^{\pm 1}, b^{\pm 1}\}$, then Theorem 3.6 proves that the set of words with more b's than a's is ω-absorbing if ν puts most of its mass on $\{b^{\pm 1}\}$ but not ω-absorbing if ν puts most of its mass on $\{a^{\pm 1}\}$. This does not complete the paper because (being nonamenable) Γ has no Liouville property.

§4 explains how to construct a manifold modelling a reversible Markov chain. The main theorem is 4.3; however this on its own would not be enough to obtain all the properties the model and the original chain have in common. It is often important to exploit the symmetry of the Laplace Beltrami operator in addition to 4.3 to get sharp results.

§5 considers the manifold model of the simple random walk on Γ and shows how the arguments used in §3 when combined with 4.3 allow one to deduce the analogous theorem on the instability of ω-absorbing sets as Theorem 3.6. Again there are plenty of bounded harmonic functions.

§6 is concerned with modifying the Markov chain on Γ in a nonstationary way so as to make it strong Liouville without invalidating Theorem 3.6 which gives the instability of ω-absorbing sets. It is the most laborious part of the construction using "symmetries" of the group to simplify the presentation.

§7 goes through the by now routine extension argument to turn the instability of ω-absorbing sets for Liouville Markov chains into an instability of the Liouville property. The final conclusion is that there are two quasi-equivalent chains—one has only constant positive harmonic functions, the other has an exactly two-dimensional cone of positive harmonic functions all of which are bounded. Thus the example is not even weakly Liouville.

§8 carries out analogous arguments to those in §§6 and 7 for the manifold examples. The same conclusions hold, however there are differences in the details.

1. Quasi-equivalent reversible Markov chains

Let X be a countable set and $\mathbf{q} = (q_{ij})_{i,j \in X}$ be a symmetric positive function on $X \times X$, zero on the diagonal and with $\pi_i = \sum_{j \in X} q_{ij} < \infty$ for all i. Because $\mathbf{q}$ is symmetric it determines a special type of Markov chain on X

called a *reversible Markov chain*. This chain can either be viewed in continuous time in which case the q_{ij} is the Poisson rate of jumping from i to j and π_i is the depletion rate from site i, or it can be viewed in discrete time in which case $p_{ij} = q_{ij}/\pi_i$ is the probability that the next jump will be to site j given that the process is currently at i. The connection between the two processes is that the discrete process records precisely the jumps of the continuous time process. Reversibility is significant because in this case alone the infinitesimal generator of the Markov chain is symmetric. Henceforth and without further mention all our Markov chains will be reversible.

The Laplace-Beltrami operator on a Riemannian manifold is also symmetric. The discrete and continuous set-ups determine Dirichlet spaces and these spaces are fairly similar. The similarity between the two does not guarantee all that much, however it is often true that properties of Brownian motion on manifolds can be mimicked on reversible Markov chains and vice versa [9], [14], [5].

With this in mind we proceed as follows. We say $\mathbf{q}$, $\mathbf{q}'$ determine quasi-equivalent reversible Markov chains on X if for some $C > 1$ we have $q'_{ij}/C \leqslant q_{ij} \leqslant Cq'_{ij}$ for all $i, j \in X$. We will give two examples of pairs of quasi-equivalent Markov chains, one of each pair admits nonconstant bounded harmonic functions, the other does not. (A harmonic function on X is any function f with the property $\pi_i f_i - \sum_{j \neq i} q_{ij} f_j \equiv 0$, in other words f composed with the discrete time Markov chain gives a martingale.) The first pair is relatively simple to describe but does not seem to have a manifold analogue. The second example has common features with the first example but is altogether harder. However, it does allow a manifold analogue and because one can say things about positive harmonic functions in this case, it is a stronger Markov chain example.

2. The simple Markov chain example

Our construction has two parts. Our final state space will be $\mathbb{Z} \times \mathbb{Z} \times C_2$ (where $C_2 = \{0, 1\}$ is the cyclic group of order 2) but initially we consider only $\mathbb{Z} \times \mathbb{Z}$. We define quasi-equivalent rate matrices $\mathbf{q}^\lambda$ as follows: $\mathbf{q}^\lambda_{\mathbf{ij}} = 0$ unless $\mathbf{i}, \mathbf{j} \in \mathbb{Z} \times \mathbb{Z}$ are nearest neighbors in the geometric sense. Let $\mathbf{j} = (r, s)$; then

$$q^\lambda_{(r,s-1),(r,s)} = \lambda 2^{r+s} \quad \text{and} \quad q^\lambda_{(r-1,s),(r,s)} = 2^{r+s},$$

all other terms being determined by symmetry (see for example Figure 1).

Now translations do not preserve $\mathbf{q}^\lambda$, rather they multiply all the rates by a fixed power of 2. Since the transition probabilities $p^\lambda_{ij} = q^\lambda_{ij}/\pi_i$ are not affected by such transformations we see that the Markov chain (in discrete time) is invariant under the group action. Let $\nu_\lambda(\mathbf{j}) = p^\lambda_{(0,0),\mathbf{j}}$. We see that h is harmonic on $\mathbb{Z} \times \mathbb{Z}$ for $\mathbf{q}^\lambda$ if and only if $\nu_\lambda * h = h$. As $\mathbb{Z} \times \mathbb{Z}$ is abelian the classical theorem of Choquet and Deny [3] implies that if h is bounded and harmonic, then h is constant. So we have the weak Liouville property:

Proposition 2.1. $(\mathbb{Z} \times \mathbb{Z}, \mathbf{q}^\lambda)$ *admits only constant bounded harmonic functions.*

Let X_i be independent random variables with values in $\mathbb{Z} \times \mathbb{Z}$ and law ν_λ. One observes that the random walk from $(0,0)$ determined by $\mathbf{q}^\lambda$ can be realised as $Z_n = \sum_1^n X_i$. If $\mu_\lambda = \mathbb{E}(X_1)$, then by the strong law of large numbers [2] $n^{-1}Z_n$ converges to μ_λ almost surely. In particular, one observes that if $\lambda > 1$, then Z_n eventually remains strictly above the diagonal $\{(r,r) \mid r \in \mathbb{Z}\}$ in $\mathbb{Z} \times \mathbb{Z}$, and if $\lambda < 1$, then it remains strictly below. We will require the following rather weak consequence.

Proposition 2.2. *There is a subset $A \subset \mathbb{Z} \times \mathbb{Z}$ and two choices $\lambda, \lambda' > 0$ so that the $\mathbf{q}^\lambda$ process finally enters A and remains there with probability one and the $\mathbf{q}^{\lambda'}$ process quits A without ever returning to it—again with probability one. We will say A is ω-absorbing for the $\mathbf{q}^\lambda$ process and ω-transient for the $q^{\lambda'}$-process.*

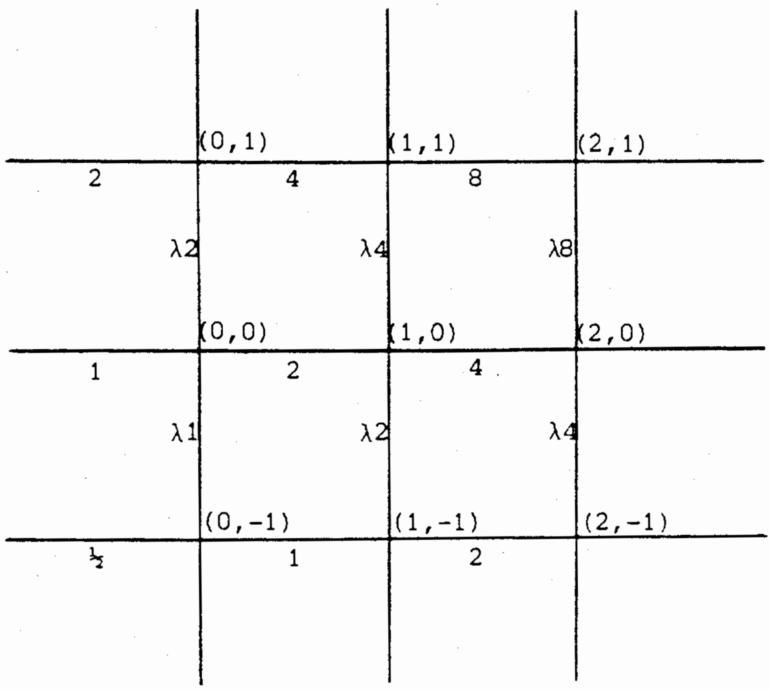

FIGURE 1

We now explain a simple construction which we use again later in the more sophisticated example. Take the set $\mathbb{Z} \times \mathbb{Z} \times C_2$, where $C_2 = \{0, 1\}$, and define two symmetric rates on the set so that the $\mathbb{Z} \times \mathbb{Z}$ coordinate Z_t of the random walk in continuous time (Z_t, W_t) is independent of the C_2 coordinate W_t and looks exactly like the q^λ ($q^{\lambda'}$)-process defined above. The C_2 coordinate W_t alternates at a positive rate only when Z_t is in A, and then at a rate comparable to the jumping rates prevailing in $\mathbb{Z} \times \mathbb{Z}$ at Z_t. To be precise, if $\mathbf{m}, \mathbf{n} \in \mathbb{Z} \times \mathbb{Z}$, $i = 0$ or 1, then

$$\rho_{(\mathbf{m},i),(\mathbf{n},i)} = q^\lambda_{(\mathbf{m},\mathbf{n})}, \qquad \rho'_{(\mathbf{m},i),(\mathbf{n},i)} = q^{\lambda'}_{(\mathbf{m},\mathbf{n})},$$

and if $\mathbf{m} \in A$ we also put

$$\rho_{(\mathbf{m},0),(\mathbf{m},1)} = \sum_{\mathbf{n} \in \mathbb{Z} \times \mathbb{Z}} q^\lambda_{\mathbf{m},\mathbf{n}} = \pi_{\mathbf{m}}, \qquad \rho'_{(\mathbf{m},0),(\mathbf{m},1)} = \sum_{\mathbf{n} \in \mathbb{Z} \times \mathbb{Z}} q^{\lambda'}_{\mathbf{m},\mathbf{n}} = \pi'_{\mathbf{m}},$$

otherwise the rates are zero. We have the following.

Theorem 2.3. $(\mathbb{Z} \times \mathbb{Z} \times C_2, \rho)$ *admits no bounded nonconstant harmonic function.* $(\mathbb{Z} \times \mathbb{Z} \times C_2, \rho')$ *admits a two-dimensional space of bounded harmonic functions.*

Proof. Let h be a positive bounded harmonic function on $(\mathbb{Z} \times \mathbb{Z} \times C_2, \rho)$ and let $\tilde{h}$ denote its reflection in the C_2 coordinate. This is also harmonic and $(h + \tilde{h})/2$ is a constant c because it is C_2 invariant and so induces a bounded harmonic function on $(\mathbb{Z} \times \mathbb{Z}, q^\lambda)$. It follows that if $h((\mathbf{m}, 0)) > c$, then $h(\mathbf{m}, 1) < c$, etc. If the random walk X_t induced by ρ has coordinates (Z_t, W_t), then with probability one W_t changes value infinitely many times as $t \to \infty$ because A is ω-absorbing for Z_t. Therefore $\limsup_{t\to\infty} h(X_t) \geqslant c$ and $\liminf_{t\to\infty} h(X_t) \leqslant c$. But $h(X_t)$ is a bounded martingale, hence its limit exists almost surely and $h(X_t)$ can be recovered from this limit by conditional expectation. Thus $h(X_t) = c$ with probability one and so h is constant.

We now consider the process $X_t' = (Z_t', W_t')$ induced by ρ'. We note that, in contrast with the previous case, with probability one W_t' changes value only finitely many times as t increases to infinity (because A is transient for Z_t'). It follows that if $h_0(x) = \mathbb{P}\,(W_t'$ is eventually $0 \mid W_0 = x)$, then h_0 is a nonconstant bounded harmonic function on $\mathbb{Z} \times \mathbb{Z} \times C_2$; $h_1 = 1 - h_0$ is another.

Let f be bounded and harmonic so that $0 < f < c(h_0 + h_1)$; it follows from the lattice property [4] for potential theory (or the martingale convergence theorem since h_0, h_1 are bounded) that $f = f_1 + f_2$, where $f_i < ch_i$, $i = 0, 1$ and f_i is harmonic. Because of the particular choice of h_0, h_1 here we see that the decomposition of f is unique. Abstractly this is because the measures on the Martin boundary giving rise to h_0, h_1 are mutually singular. A simple proof in our situation comes from the following observation. Any bounded of the remaining Brownian paths to terminate on the open end of C_i. On each occasion we put approximately the same proportions of the existing paths on the open ends of the tubes $(C_i)_{i=1}^r$. Eventually all paths will be included in this iteration so we see that the probabilities of exit from the various tubes must also be in these approximate proportions:

$$\mathbb{P}(z_t \text{ exits } S(d_1,\cdots,d_r) \text{ through } C_i)$$
$$\leqslant \frac{(1+\varepsilon)^2/d_i}{(1+\varepsilon)^{-2}\Sigma_{k=1}^r(d_k)^{-1}} = \frac{(1+\varepsilon)^4/d_i}{\Sigma_{k=1}^r(d_k)^{-1}}$$

and is also

$$\geqslant \frac{(1+\varepsilon)^{-2}/d_i}{(1+\varepsilon)^2\Sigma_{k=1}^r(d_k)^{-1}}.$$

This is what we required.

5. A manifold model of the free group Γ and an extension of Theorem 3.6

Let Q be a compact Riemann surface with four discs excised from it and let $S(R/p, R/p, R/q, R/q)$ be the surface obtained by adjoining four tubes which on this occasion we label $C_{a^{\pm 1}}$, $C_{b^{\pm 1}}$. Now take Γ copies $(S_g)_{g\in\Gamma}$ of $S(R/p, R/p, R/q, R/q)$ (where $p+q=1$) and let $\sim$ be the equivalence relation identifying $C_{a^{-1}}$ in S_g with C_a in $S_{a^{-1}g}$, $C_{b^{-1}}$ in S_g with C_b in $S_{b^{-1}g}$, etc. Then $M = \bigcup_{g\in\Gamma} S_g$ is a Riemann surface without boundary with Γ acting as a discontinuous group of isometries on it by $h\in\Gamma$ taking S_g onto S_{gh}. M/Γ is a compact manifold obtained by adjoining two more handles to Q.

As in §4, let z_t be Brownian motion on M, and let s_n be the nth occasion that z_t emerges from one of the tubes $C_{g^{\pm 1}}$, $g\in\{a,b\}$, having last entered it from the other end. At that time s_n, the Brownian path z_t is at an interior point of exactly one S_g, $g\in\Gamma$; let $\tilde{X}_n$ be that g. We have the following immediate consequence of Theorem 4.3.

Proposition 5.1. *Fix $\varepsilon > 0$. Then there is an R such that for any sequence $\tilde{X}_0,\cdots,\tilde{X}_{n-1}$ we have the uniform estimates*

$$\frac{p}{2(1+\varepsilon)} < \mathbb{P}\left(\tilde{X}_n\tilde{X}_{n-1}^{-1} = a^i \mid X_{n-1},\cdots,X_0\right) < \frac{p(1+\varepsilon)}{2},$$
$$\frac{q}{2(1+\varepsilon)} < \mathbb{P}\left(\tilde{X}_n\tilde{X}_{n-1}^{-1} = b^i \mid X_{n-1},\cdots,X_0\right) < \frac{q(1+\varepsilon)}{2}$$

for all choices of p and $i = \pm 1$.

So, as before, although $\tilde{X}$ is not Markov, in this quite precise sense it is nearly so. Sufficiently nearly so that the arguments in §3 may be repeated. Let t_N be the Nth occasion that $\tilde{X}_n\tilde{X}_{n-1}^{-1} = a^{\pm 1}$, and consider the term $\tilde{g}_N = \tilde{X}_{t_n}\tilde{X}_{t_{n-1}}^{-1} = a^{\tilde{\sigma}_n}b^{\tilde{\tau}_n}$. If we let $p_1 = (1+\varepsilon)p$ and $p_2 = (1-\varepsilon)p$ we have the following estimates.

Theorem 5.2.

$$p_2(1-p_1)^i \leqslant \mathbb{P}(t_{N+1} - t_N = i+1 \mid X_n,\, n \leqslant t_N) \leqslant p_1(1-p_2)^i,$$

$$\sum_{i=0}^{\infty} p_2(1-p_1)^{j+2i}\frac{2^{-(j+2i)}(j+2i)!}{i!(j+i)!}$$

$$\leqslant \mathbb{P}(\tilde{\tau}_N = j \mid X_n,\, n \leqslant t_N) \leqslant \sum_{i=0}^{\infty} p_1(1-p_2)^{j+2i}\frac{2^{-(j+2i)}(j+2i)!}{i!(j+1)!},$$

where $i \geqslant 0$, $i, j \in \mathbb{Z}$.

From these estimates it is trivial that if p_0 is chosen as in Theorem 3.6 and p is less than $p_0(1+\varepsilon)^{-1}$, then the arguments and estimates in the proof of Theorem 3.6 apply equally here.

6. A nonstationary random walk on Γ; the elimination of positive harmonic functions

In §3 we introduced a parametrized family of random walks on Γ with the significant property that Γ admitted subsets which were ω-absorbing for one choice of parameter but not for others. However any stationary random walk on a nonamenable group (such as Γ) will admit nonconstant bounded harmonic functions. So the idea of constructing an example by joining two copies as in §2 will not work. In this section we will describe nonspatially homogeneous modifications to the random walks on Γ introduced in §3. The resulting Markov chains retain the property concerning ω-absorption but do not admit positive harmonic functions. Later sections will extend these modifications to the manifold model and construct the final example.

In [9] a sufficient condition was given for a positive harmonic function on a manifold to be completely determined by its values at a particular sequence of points x_n. Although it was not explicitly pointed out there, the sequence did not need to consist of distinct points; letting them all coincide, one has the following condition which is sufficient (and obviously necessary) for the nonexistence of positive harmonic functions.

Let (X, p_{ij}) be any Markov chain and let $A_i \subset X$ be a sequence of subsets of X with $\bigcup A_i = X$. Let $B_i \supset A_i$ be a second sequence of sets with the property $\sum_{k\in B_i} p_{jk} = 1$ whenever $j \in A_i$. We will talk about a numerical function f being harmonic on B_i if $\sum_{k \in X} p_{jk} f(k) = f(j)$ whenever $j \in B_i$. We may then consider the Harnack constant c_i of the pair (A_i, B_i). That is

$$c_i = \sup\left\{ \frac{h(x)}{h(y)} \,\middle|\, x, y \in A_i,\ h \text{ positive on } X \text{ and harmonic on } B_i \right\}.$$

We note that in any situation where A_i is finite and communication in B_i between any two sites of A_i is possible, then c_i is finite.

Lemma 6.1 [9]. *Let x_i be a sequence in X. Any positive harmonic function on X is uniquely determined by its values on (x_i) if there is a choice of pairs $(A_i \subset B_i)$ with $x_i \in A_i$, $\bigcup A_i = X$, and a uniform bound C on the Harnack constants c_i of the pairs A_i, B_i.*

In particular if $\bigcap_i A_i$ is nonempty, then we deduce from Lemma 6.1 that all positive harmonic functions are constant. (The referee has pointed out that in this special case one has a simpler argument because $\sup h \leqslant c \inf h$ and if h is nonconstant we may assume the right-hand side is zero.) We now explain in detail the modified walks on Γ.

Let θ_n be a permutation of the $4 \cdot 3^{n-1}$ reduced words of length n. Choose the permutation so that it is of order 3^{n-1} with our equal orbits: the reduced words commencing on the left with a, a^{-1}, b, and b^{-1} respectively. Let $g \in \Gamma$ be of reduced length at least n; there is a unique factorization of g into $g_1 g_2$ with $|g_1| + |g_2| = |g|$, $|g_2| = n$. Define $g\theta_n = g_1(g_2\theta_n)$. Then because θ_n does not alter the leftmost element of g_2 we see that $|g\theta_n| = |g|$ and so we have extended θ_n to be a permutation of the words of length m for each $m \geqslant n$. With a bit of thought the sequence $(\theta_n)_{n=1}$ can be arranged so that $(\theta_n)^3 = \theta_{n-1}$ whenever both are defined. The essence of θ_n is that if we apply it to an element g with $|g| \gg n$, then $g\theta$ and g have essentially the same numbers of a's and b's etc.

Define symmetric rates $\mathbf{r}(p)$ on Γ as follows

$$r(p;\, g_1, g_2) = \begin{cases} \dfrac{p}{2} & \text{if } g_1 g_2^{-1} = a^{\pm 1}, \\ \dfrac{1-p}{2} & \text{if } g_1 g_2^{-1} = b^{\pm 1}, \\ 1 & \text{if (i) } g_1 = g_2\theta_n^{\pm 1} \text{ and (ii) } |g_1| \in [R_n, S_n], \\ 0 & \text{otherwise,} \end{cases}$$

where R_n, S_n are fixed integers satisfying $n < R_n < S_n < R_{n+1}$ for each n. For suitable choice of R_n, S_n this will have all the required properties. The evolution of $(X', \mathbf{r}, \Gamma)$ is best understood in continuous time. X' evolves by being left multiplied by $a^{\pm 1}$'s and $b^{\pm 1}$'s at rates $p/2$, $q/2$ and occasionally (when the length of the word lies in one of the bands $[R_n, S_n]$) the rightmost n

letters of X' are modified at rate 1. We may couple X' with the stationary random walk introduced in §3 by letting $X_t = (X'_{t_n} \cdot X'^{-1}_{t_n^-}) \cdots (X'_{t_1} \cdot X'^{-1}_{t_1^-})$, where t denotes the ith occasion that X' is modified on the left and n is $\max\{i \mid t_i < t\}$.

We will now prove that providing the R_n, S_n satisfy $\lim_{n\to\infty} n^{-1}R_n = \infty$, then

$$\lim_{t\to\infty} \frac{l_a(X'_t)}{l_a(X_t)} = \lim_{t\to\infty} \frac{l_b(X'_t)}{l_b(X_t)} = 1.$$

In particular we see that as p varies, the set of $\{g \mid l_a(g) > l_b(g)\}$ can be made either ω-absorbing or the complement of such a set.

First observe that $l(X'_t)/t$ tends to a strictly positive limit for each choice of p. In particular with probability one there will be a last time T_ε that $\inf_{s>t}(|X'_s|/|X'_t|) < 1 - \varepsilon$. From this time onward simple algebra allows one to estimate $|X_t^{-1}X'_t|$ as follows.

Lemma 6.2. *For any $t > T_\varepsilon$ we have*

$$|X_t^{-1}X'_t| \leqslant |X_{T_\varepsilon}| + |X'_{T_\varepsilon}| + 2\max\{k \mid \exists s \in [T_\varepsilon, t),\ X'_s \in [R_k, S_k]\}.$$

Proof. After t exceeds T_ε it is clear that never again will $|X'_t|$ go below k after having exceeded R_k. In particular a simple induction argument shows that X'_t and $X_t X_{T_\varepsilon}^{-1} X'_{T_\varepsilon}$ differ only in the last K terms, where $K = \max\{k \mid \exists s \in [T_\varepsilon, t),\ X'_s \in [R_k, S_k]\}$, so $|X'^{-1}_{T_\varepsilon} X_{T_\varepsilon} X_t^{-1} X'_t| \leqslant 2K$ and the result follows.

Remark. As $R_k/k \to \infty$ we see that

$$|X_t^{-1}X'_t| \leqslant |X_{T_\varepsilon}| + |X'_{T_\varepsilon}| + \frac{2|X'_t|K}{(1-\varepsilon)R_K} \quad \text{for all } t > T_\varepsilon.$$

In particular as t goes to ∞, K goes to infinity and $|X_t^{-1}X'_t| = o(|X'_t|)$. Because the R_n, S_n do not depend on p we have shown that the remarks of §3 carry over to the Markov chains X'_t obtained as p varies.

Now, using Lemma 6.1 we wish to prove that providing $S_n - R_n$ grows more rapidly than $(3^{n-1})^{2+\varepsilon}$ for some $\varepsilon > 0$, then $(\Gamma, \mathbf{r}(p))$ does not admit positive harmonic functions. Our arguments in the Markov chain and Manifold cases are not the same; in the latter case we obtain an existential theorem for each fixed p (or finite selection of p's) but no estimate of growth of rate of $S_n - R_n$; in the Markov chain case one obtains much sharper results. The main problem is to eliminate positive harmonic functions—bounded harmonic functions are easier.

The main technical result will be the following:

Proposition 6.3. *If* $|\gamma| = R_n$ *and* $T = \inf\{t \mid |X_t'| \geqslant S_n\}$, *then*

$$\forall j \leqslant 3^{n-1}, \forall A \subset \{\xi \mid |\xi| = S_n\},$$

$$\frac{1}{C} < \frac{\mathbb{P}^{\gamma}\left(X_T' \in A \mid |X_t'| \geqslant R_n, \forall t < T\right)}{\mathbb{P}^{\gamma}\left(X_T' \in A\theta_n^j \mid |X_t'| \geqslant R_n \, \forall t < T\right)} \leqslant C.$$

Remark. The theorem says that translating A by θ_n does not affect the hitting probabilities. This is not surprising because, conditioned on $|X_t'|$ never going below R_n, we may decouple the X and θ actions completely and make them independent.

Proof. Let Θ_t be a continuous time random walk on the cyclic group of permutations $\langle \theta_n^j \mid j \leqslant 3^{n-1} \rangle$ jumping from $\theta_n^j \to \theta_n^{j \pm 1}$ with rate one, and let X_t be chosen independently to be the simple random walk on Γ determined in §3 with $X_0 = \gamma$ and $|\gamma| = R_n$ but conditioned so that $|X_t| \geqslant \gamma$, $\forall s < T$. Then we have $X_t\Theta_t$ is identical in law to X_t' conditioned so that $|X_s'| \geqslant \gamma$ for all $s < T$. This separation of X_t' into two parts, together with the approximate equidistribution, will yield the theorem.

For general times s the two variables X_s, Θ_s are not independent, however if $t_1 < t_2 < \cdots < t_{S_n - R_n} < \cdots < t_k$ denote the successive jump times of X_t, then $(\Theta_{t_k})_{k \in \mathbb{N}}$ and $(X_{t_k})_{k \in \mathbb{N}}$ are independent Markov chains for $t_k < T$. So

$$\mathbb{P}^{\gamma}(X_T' \in A) = \sum_{k \geqslant S_n - R_n} \sum_{0 \leqslant j < 3^{n-1}} \mathbb{P}\left(X_{t_k} \in A\theta_n^{-j}, t_k = T\right) \cdot \mathbb{P}\left(\Theta_{t_k} = \theta_n^j\right).$$

But as $S_n - R_n > (3^{n-1})^{2+\varepsilon}$ and t_k is always greater than $S_n - R_n$ it follows that Θ_{t_k} is approximately equidistributed and

$$\mathbb{P}\left(\Theta_{t_k} = \theta_n^j\right) \sim 3^{-(n-1)}.$$

So

$$\mathbb{P}^{\gamma}(X_T' \in A) \sim \frac{1}{3^{n-1}} \sum_{0 \leqslant j < 3^{n-1}} \mathbb{P}\left(X_T \in A\theta_n^{-j}\right)$$

and the right-hand side is invariant under replacing A by $A\theta_n^{-j'}$. So the lemma is proved modulo the claim that the random walk on the cyclic group with 3^{n-1} elements $\langle \theta_n^j \rangle$ given by Θ_{t_k} is approximately equidistributed by the time $k > S_n - R_n > (3^{n-1})^{2+\varepsilon}$.

This can be shown with an explicit calculation using the probability generating function of a geometric mixture of balanced binomial distributions or by using more general central limit type theorems.

As we are unable to use such sharp techniques or to obtain the rate of convergence in the manifold case we do not give further details of the computation here. Rather we explain the significance of Proposition 6.3.

Let us now suppose that $|\gamma_0| \leqslant n$; we run the unconditioned X_t' from $\gamma_0 = X_0'$ until T; the first time $|X_t'| > S_n$. We wish to compute $\mathbb{P}^{\gamma}(X_T' \in A)$ and show that it does not depend much on the choice of γ_0. This is achieved by splitting the walk $(X_t')_{0<t<T}$ into three parts. Let τ_1 denote the time X_t' last exits $\{|\gamma| < n\}$ and τ_2 the time of last exit from $\{|\gamma| < R_n\}$. Both of these times are splitting (but not stopping) times so $X'_{t+\tau_1}$, $X'_{t+\tau_2}$ are both Markov processes—in fact $(X'_{t+\tau_i})_{t=0}^{t=T-\tau_i}$ is identical in distribution to X_t' started on $\{|\gamma| = n \text{ (or } R_n)\}$ but conditioned to remain in $\{|\gamma| \geqslant n \text{ (or } R_n)\}$. Moreover, conditional on X'_{τ_1}, the process $X'_{t+\tau_1}$ is independent of X_i', $t < \tau_1$. So we have

$$\mathbb{P}^{\gamma}\big(X_T' \in A\big) = \sum_{|\gamma_1|=n} \mathbb{P}^{\gamma}\big(X'_{\tau_1} = \gamma_1\big)\mathbb{P}^{\gamma_1}\big(X_T' \in A|\ |X_s| \geqslant n, \forall s < T\big).$$

But we claim that $\mathbb{P}^{\gamma_1}(X_T' \in A||X_s'| \geqslant n, \forall s < T)$ is essentially (to within a fixed factor independent of A, n) independent of the choice of γ_1. To see this we repeat the splitting argument and condition on X_{τ_2}:

$$\begin{aligned}\mathbb{P}^{\gamma_1}&\big(X_T' \in A|\,|X_s'| \geqslant n, \forall s < T\big)\\ &= \sum_{|\gamma_2|=R_n} \mathbb{P}^{\gamma_1}\big(X'_{\tau_2} = \gamma_2\big)\cdot\mathbb{P}^{\gamma_2}\big(X_T' \in A|\,|X_s'| \geqslant R_n, \forall s < T\big)\end{aligned}$$

and by Proposition 6.2 this equals (to within fixed small multiples)

$$\sum_{|\gamma_2|=R_n} \mathbb{P}^{\gamma_1}\big(X'_{\tau_2} = \gamma_2\big)\mathbb{P}^{\gamma_2}\big(X_T' \in A\theta_n^j|\,|X_s'| \geqslant R_n, \forall s < T\big), \quad \forall j < 3^{n-1},$$

but the random walk on $\{\gamma||\gamma| \geqslant n\}$ is preserved by θ_n^j (because θ_n^j was chosen so that $(\theta_n^j)^3 = \theta_{n-1}^j$) so this last expression equals

$$\sum_{|\gamma_2|=R_n} \mathbb{P}^{\gamma_1}\big(X'_{\tau_2} = \gamma_2\big)\mathbb{P}^{\gamma_2\theta_n^{-j}}\big(X_T' \in A|\,|X_s'| \geqslant R_n, \forall s < T\big).$$

Using the θ_n^j invariance again,

$$\mathbb{P}^{\gamma_1}\big(X'_{\tau_2} = \gamma_2\big) = \mathbb{P}^{\gamma_1\theta_n^{-j}}\big(X'_{\tau_2} = \gamma_2\theta_n^{-j}\big),$$

and contracting the sum again we have

$$\begin{aligned}&\mathbb{P}^{\gamma_1}\big(X_T' \in A|\,|X'| \geqslant n, \forall s < T\big)\\ &\qquad\sim \mathbb{P}^{\gamma_1\theta_n^{-j}}\big(X_T' \in A|\,|X_s'| \geqslant n, \forall s < T\big), \quad \forall j < 3^{n-1}.\end{aligned}$$

Let $\mu_1 \cdots \mu_4 \in \Gamma$ be representatives of the four orbits $O_1, \cdots, O_4$ determined by the θ_n action on the words of length n. We have

$$\mathbb{P}^{\gamma_0}\big(X_T' \in A\big) \sim \sum_1^4 \mathbb{P}^{\gamma_0}\big(X'_{\tau_1} \in O_i\big)\mathbb{P}^{\mu_i}\big(X_T' \in A|\,|X_s| \geqslant n, \forall s \leqslant t\big).$$

Consider $\mathbb{P}^{\gamma_0}(X_{\tau_1} \in O_i)$. There is a lower bound c depending only on p for $\mathbb{P}(|X_t'| \geqslant |X_0'|, \forall t > 0)$ which is independent of the position of X_0. It follows that on $|\gamma_0| = n$ we have $\mathbb{P}^{\gamma_0}(X_{\tau_1} \in O_i) > c_1$, $\forall \gamma_0 \in O_i$; moreover $\mathbb{P}^{\gamma_0}(X_{\tau_1} \in O_i)$ is harmonic on $|\gamma_0| < n$ and since γ_0 is at most a word of length 4 away from any of the O_i if $|\gamma_0| = n - 1$, there is a second constant c_2 such that $\mathbb{P}^{\gamma_0}(X$ quits $|\gamma| < n$ through $O_i) > c_2$, $|\gamma_0| = n - 1$; it follows that $\mathbb{P}^{\gamma}(X_{\tau_1} \in O_i) > c_1 c_2$, $\forall \gamma$ with $|\gamma| = n - 1$ and hence by the minimum principle for all γ with $|\gamma| < n$.

Combining $\mathbb{P}^{\gamma_0}(X_{\tau_1} \in O_i) \sim 1$ with the expression for $\mathbb{P}^{\gamma_0}(X_T' \in A)$ above we have

$$\mathbb{P}^{\gamma_0}\left(X_T' \in A\right) \sim \sum_1^4 \mathbb{P}^{\mu_i}\left(X_T' \in A \,|\, |X_s| \geqslant n, \forall s \leqslant t\right)$$

for all γ_0 with $|\gamma_0| < n$. The right-hand side does not depend on γ_0 so we have proved:

Theorem 6.4. *There is an absolute constant C depending only on the choice of $(R_n, S_n)_{n=1}^{\infty}$ such that if $A \subset \{\gamma \in \Gamma \,|\, |\gamma| = S_n\}$ and $T = \inf_s\{|X_s'| \geqslant S_n\}$, then*

$$\sup_{|\gamma| < n} \mathbb{P}^{\gamma}\left(X_T' \in A\right) \leqslant C \inf_{|\gamma| < n} \mathbb{P}^{\gamma}\left(X_T' \in A\right).$$

Remark. It follows that if $h > 0$ is harmonic on $|\gamma| \leqslant S_n$, then

$$\sup_{\substack{\gamma, \gamma' \in \Gamma \\ |\gamma|, |\gamma'| < n}} \left[\frac{h(\gamma)}{h(\gamma')}\right] < C.$$

Thus by Lemma 6.1 and the remark following it, Γ with the new random walk X_t' admits no positive harmonic functions.

7. Completing the Markov chain example

§6 was dedicated to constructing a one-parameter family of reversible Markov chains X' on Γ with the following properties.

(i) For any choice of the parameter p the random walk on Γ admits no global positive harmonic functions other than constants.

(ii) For one choice of p it follows that $l_a(X_t') > l_b(X_t')$ for all large t with probability one, for another $l_b(X_b') > l_a(X_t')$.

(iii) There is a uniform upper bound (six) on the number of sites in Γ one can jump to from a given site. Further the rates of jumping to these sites given by $\mathbf{r}(p)$ are also uniformly bounded.

We now proceed as in §2. Let $\Pi = \Gamma \times \{0,1\}$, and give it reversible Markov chain structure by defining $\pi(p)$ by

$$\pi\big(p;\,(\gamma,s),(\gamma',s)\big) = r(p;\,\gamma,\gamma'), \qquad \gamma,\gamma' \in \Gamma, \forall s \in \{0,1\},$$

and

$$\pi\big(p;\,(\gamma,0),(\gamma,1)\big) = \pi\big(p;\,(\gamma,1),(\gamma,0)\big) = 1 \quad \text{if } l_a(\gamma) > l_b(\gamma).$$

We extend π to $\Pi \times \Pi$ by making it zero if not otherwise specified as above.

The Γ coordinate of this new process Y_t on Π is indistinguishable from X_t'. So for $p = p_1$ close to 1 the process Y_t eventually stays in $\{l_a(\gamma) > l_b(\gamma)\} \times \{0,1\}$ and so crosses from one copy of Γ to the other infinitely many times with probability one; if p is small, Y_t will stay on one copy of Γ or the other for large t. We prove our main theorem.

Theorem 7.1. $(\Pi, \pi(p_1))$ *admits no nonconstant positive harmonic functions,* $(\Pi, \pi(p_2))$ *admits precisely a two-dimensional cone of positive harmonic functions, and all positive harmonic functions are bounded.*

Proof. For any p, we note that if h is $\pi(p)$ harmonic and h^r denotes its reflection (i.e., $h^r((\gamma,0)) = h((\gamma,1))$, etc.), then h^r is also harmonic. Moreover $h + h^r$ is constant because $(\Gamma, \mathbf{r}(p))$ admitted no positive harmonic functions except constants. So we always have $h(\gamma,0) + h(\gamma,1) = c$ independent of $\gamma \in \Gamma$. The positive harmonic functions on Π form a convex cone, and when normalized to be one at any fixed $x_0 \in \Pi$ one obtains a compact set because the Harnack conditions alluded to before 6.1 give equi-continuity and uniform boundedness on bounded sets. It follows that every positive harmonic function is a convex combination of extremal (or minimal) positive harmonic functions (that is, functions on the extreme Martin boundary). Such an extremal function h has the property that if $h_1 > 0$ is harmonic with $\alpha h_1 < h$, $\alpha > 0$, then $\beta h_1 = h$ for some β. It is a simple consequence of this that if h is extremal harmonic, then either (i) $\lim_{t\to\infty} h(Y_t') = 0$ for almost all w and for all starting points $x = Y_0$, or (ii) $\mathbb{P}^x(\lim_{t\to\infty} h(Y_t') \in \{0,c\}) = 1$ for all x and some c. Let

$$h_1(x) = \mathbb{E}^x\Big(\lim_{t\to\infty} h(Y_t) \cdot X\Big\{\lim_{s\to\infty} h(Y_t) \in (a,b)\Big\}\Big).$$

Then $0 \leqslant h_1 \leqslant h$, h_1 is harmonic, and $\lim_{t\to\infty} h(Y_t)/h_1(Y_t) = 1$ if $\lim_{t\to\infty} Y_t \in (a,b)$, $a > 0$. By varying a, b we see that if h is unbounded and minimal then (i) applies. If h is bounded and minimal then (ii) applies.

We will now show that for any p, any minimal positive harmonic function h on $[\Pi, \pi(p)]$ must be bounded. Let Y_t be the random walk, and let $\tilde{Y}_t$ denote its reflection. Then $\tilde{Y}_t$ is also a Markov chain determined by $\pi(p)$. If h is unbounded then (i) implies

$$\lim_{t\to\infty}\left[h(Y_t)+h(\tilde{Y}_t)\right]=0,$$

but $h(\gamma,0)+h(\gamma,1)$ is constant independent of γ so h must be zero—a contradiction. Thus all minimal harmonic functions are bounded.

We now consider the cases p small and p close to 1. The latter is easy; let h be a bounded minimal harmonic function normalized so that $h(\gamma,0)+h(\gamma,1)=2$. Then because Y_t crosses between $\Gamma\times\{0\}$ and $\Gamma\times\{1\}$ at arbitrarily large times and at these times $h(Y_t)$ crosses the value 1 we have with $\mathbb{P}^x$ probability one for all x

$$\lim_{t\to\infty} h(Y_t)=1.$$

But $h(x)=\mathbb{E}^x(\lim_{t\to\infty}h(Y_t))$; so h is constant as required.

Let p be small and $h_1(x)$ be defined to be $\mathbb{P}^x(\lim_{t\to\infty}Y_t\in\Gamma\times\{0\})$ and $h_2=h_1^r$, so that $h_1+h_2=1$. As in §2.3 a standard probabilistic or lattice type argument shows that any positive harmonic function with $\|h\|_\infty<1$ satisfies $h=h_1'+h_2'$, where $h_i'\leqslant h_i$, $i=1,2$. If h_1, h_2 are minimal, then h would be a linear combination of them, and the theorem would be proven.

Suppose $\tilde{h}<h_1$. Then $\lim_{t\to\infty}\tilde{h}(Y_t)>0$ implies that X_t is eventually in $\Gamma\times\{0\}$ so $\lim_{t\to\infty}\tilde{h}^r(Y_t)=0$. But $\tilde{h}+\tilde{h}^r\equiv c$ so it follows that with probability one either $\lim_{t\to\infty}\tilde{h}(Y_t)=c$ and $\lim_{t\to\infty}\tilde{h}^r(Y_t)=0$ or $\lim_{t\to\infty}\tilde{h}^r(Y_t)=c$ and $\lim_{t\to\infty}h(Y_t)=0$ according to whether Y_t is eventually in $\Gamma\times\{0\}$ or not. Identifying the limiting values of the two bounded martingales $\tilde{h}(Y_t)$ and $ch_1(Y_t)$ we see that $\tilde{h}=ch$, so h_1 is minimal. The argument for h_2 is identical and hence we finally have any positive harmonic function as a linear combination of h_1 and h_2 as claimed.

8. The manifold example

In this section we describe a pair of surfaces, quasi-isometrically equivalent, such that the first admits nonconstant bounded harmonic functions while the second does not even admit positive nonconstant harmonic functions. Following §4, we will build our examples along the same lines as the Markov chain examples in §§6 and 7. In those sections the reader will recall that first we had a simple translation invariant walk on Γ, then we introduced some links between the reduced words g with $|g|\in[R_n,S_n]$ enabling the process to jump from one such word to another in relatively few steps providing both such words agreed in all but the rightmost n letters. This allowed us to eradicate positive harmonic functions while leaving untouched the asymptotic ratio of a's to b's in the reduced word. Finally we took two identical copies of this modified random walk on Γ and allowed the process to flip between them if it is positioned on an element of Γ with more a's than b's. This constituted the final example. The point was that the process has two ways of getting to infinity if a's are rare compared with b's because in this case the process eventually stays on one or other copy of Γ. If a's were more common than b's in the evolution of the random walk then the process oscillates between the two copies of Γ; so there is only one way to go to infinity and the function theory is trivial.

We now introduce three manifolds L, M, N each covering a compact manifold K and such that one may identify the fibers of L, M, N over K with Γ, Γ, and $\Gamma \times C_2$ respectively. Γ will act on L so that L/Γ is K and, in the correspondence of §4, L essentially corresponds to the simple random walk, M to the modified one, and N to the double copy of Γ. N admits an isometry $\mathscr{P}$ without fixed point, consistent with the covering, such that $\mathscr{P}^2 =$ identity, and $N/\langle\mathscr{P}\rangle = M$. This corresponds to reflection. The metrics are all inherited from K. It is by varying the metric on K smoothly that we get the different behavior.

Let Q be a compact Riemann surface with eight discs excised; form S by attaching to it four pairs of cylinders $(C_{a^{\pm 1}}, C_{b^{\pm 1}}, C_{\theta^{\pm 1}}, C_{\mathscr{P}^{\pm 1}})$ with conformal radius 1 and lengths (in pairs) R/p, $R/1-p$, 1, and 1 respectively. The compact manifold K is obtained by identifying the two cylinders in each pair by overlaying them with the open end of one cylinder corresponding to the join with Q of the other. Of course there is an ambiguity concerning the angular orientation of the two cylinders but this is of no significance and we let it be fixed once and for all. In any case K is a compact Riemann surface with at least four handles, two of which we think of as variable in length.

To obtain the manifolds L, M, N we follow the reversible Markov chain constructions.

(a) L: Identify the pairs $C_{\theta^{\pm 1}}$, $C_{\mathscr{P}^{\pm 1}}$ in S and take Γ copies $[\tilde{S}(g)]_{g\in\Gamma}$ of the resulting surface $\tilde{S}$. Identify $C_a(g) \subset \tilde{S}(g)$ with $C_{a^{-1}}(ag)$ and $C_b(g)$ with $C_{b^{-1}}(bg)$. The resulting manifold L has no free cylinders and in fact has a Γ action on it such that $L/\Gamma = K$.

(b) M: Identify the pair of cylinders $C_{\mathscr{P}^\pm}$ is S to form $\tilde{\tilde{S}}$ and then take Γ copies $(\tilde{\tilde{S}}(g))_{g\in\Gamma}$ of this. Now choose $0 < R_n < S_n < R_{n+1} < \cdots$. We make identifications as follows: identify

$$\begin{array}{lll} C_a(g) \text{ with } C_{a^{-1}}(ag), & \\ C_b(g) \text{ with } C_{b^{-1}}(bg), & \\ C_\theta(g) \text{ with } C_{\theta^{-1}}(g\theta_n) & \text{if } |g| \in [R_n, S_n], \\ C_\theta(g) \text{ with } C_{\theta^{-1}}(g) & \text{if } |g| \notin \bigcup_{n\in\mathbb{N}} [R_n, S_n]. \end{array}$$

So M is obtained from L by cutting the θ handle at $\tilde{S}(g)$ whenever $|g| \in [R_n, S_n]$ and rejoining it with the corresponding parts of the handles at $S(g\theta^{\pm 1})$.

(c) N: Let $X = \Gamma \times C_2$, where Γ is generated by a, b and C_2 by c (so $c^2 = e$ and c commutes with a, b). Each x has a unique expression $x = gc^i$, $i = 0, 1$, and we extend our definitions of length, etc., to X by $|x| = |g|$, $\mathscr{A}(x) = \mathscr{A}(g)$, etc. Now take X copies $[S(x)]_{x \in X}$ of S and identify $C_a(x)$ with $C_{a^{-1}}(ax)$, $C_b(x)$ with $C_{b^{-1}}(bx)$, $G_\theta(x)$ with $C_{\theta^{-1}}(x\theta_n)$ if $|x| \in [R_n, S_n]$, $C_\theta(x)$ with $C_{\theta^{-1}}(x)$ if $|x| \notin \cup[R_n, S_n]$, and finally $C_{\mathscr{P}}(x)$ with $C_{\mathscr{P}^{-1}}(cx)$ if $\mathscr{A}(x) > \mathscr{B}(x)$ and $C_{\mathscr{P}}(x)$ with $C_{\mathscr{P}^{-1}}(x)$ if $\mathscr{A}(x) \leqslant \mathscr{B}(x)$. To obtain the symmetry $\mathscr{P}$ let $n \in S(x)$ and let n' be the element of $S(Cx)$ corresponding to n under the natural isometry of $S(x)$ and $S(cx)$. Define $\mathscr{P}(n) = n'$; it is clear that $\mathscr{P}$ is of order two, compatible with the covering of K, without fixed points, and such that $N/\langle \mathscr{P} \rangle = M$.

We have constructed L, M, N. Fix Q, R, $[R_n, S_n]_{n=1}^{\infty}$. Then as p varies we generate four 1-parameter families of homeomorphic manifolds $K(p)$, $M(p)$, $N(p)$, and $K(p)$. For each p we may give $K(p)$ the metric of constant negative curvature -1 consistent with the conformal structure and lift this up to $L(p)$, etc. If p_1, p_2 are two choices of p we may take a diffeomorphism φ of $K(p_1)$ and $K(p_2)$. Because $K(p_i)$ is compact this is a quasi-isometry and so lifts up giving quasi-isometries of $K(p_i)$, $L(p_i)$, $M(p_i)$ for $i = 1, 2$. Of course these quasi-isometries behave uniformly with respect to any local measure of distortion.

Before outlining the remaining arguments we remark that the role of R is unimportant here except that any sufficiently large R would suffice. All that is required is that Theorem 4.3 should apply with a sufficiently small ε. However it would be tedious to keep track of how small ε should be—the suspicious reader can check that only finitely many such ε will be introduced and that their values do not depend on the choice of $(R_n, S_n)_{n=1}^{\infty}$ (although they may well depend on p_1, p_2).

To complete our arguments, first we fix two values p_1 (close to zero) and p_2 (close to one) of our parameter p such that for essentially any choice of pairs $(R_n, S_n)_{n=1}^{\infty}$ the Brownian motion on $M(p_1)$ eventually stays in $\bigcup_{\mathscr{A}(g) < \mathscr{B}(g)} \tilde{S}(g)$ and for $M(p_2)$ eventually stays in the complement of this set.

Second, we choose $(R_n, S_n)_{n=1}^{\infty}$, which until now we have thought of as variables, so that $M(p_1)$ admits no positive harmonic functions for $i = 1, 2$.

Finally we show how these two properties of $M(p_i)$ are enough for one to conclude that $N(p_1)$ admits an exactly two-dimensional cone of positive harmonic functions, all of them bounded, whereas $N(p_2)$ only admits constant positive harmonic functions.

The first and third of these final arguments are mere mimics of the arguments in §§6 and 7. The second argument is different in as much as the analogue of §6.3 exists but has a different proof which (as we mentioned before) will not give a quantitative estimate for $S_n - R_n$ in terms of n. We take more care over this part than the other two. The arguments involve (in a not very deep way) the use of a boundary Harnack principle and the consideration of extremal positive harmonic functions.

Because most of our effort will be concentrated on M we introduce some extra notation here which will be helpful later. Let $r \in \mathbb{N}$ and consider all the $4 \cdot 3^{r-1}$ tubes $C_{g'g^{-1}}(g)$ which connect $\tilde{\tilde{Q}}(g)$ with $\tilde{\tilde{Q}}(g')$, where $|g| = r$, $|g'| = r - 1$, and $g'g^{-1} \in \{a^{\pm 1}, b^{\pm 1}\}$, and let D_r denote the collection of all $4 \cdot 3^{r-1}$ circles which bisect these cylinders $C_{g'g^{-1}}(g)$ into two equal cylinders of half the length. Then $M \setminus D_r$ has two components. We let $B(r)$ be the relatively compact part and $E(r)$ its complement. We let $A_{r,s} = E_r \cap B_s$ if $r < s$. We think of B_r as the ball of "radius" r, $E(r)$ as its exterior, $D(r)$ as its boundary, and $A_{r,s}$ as the annulus.

Lemma 8.1. *There are $p_1 < p_2 \in (0, 1)$ such that for all large R and essentially all choices of $(R_n, S_n)_{n=1}^{\infty}$ the Brownian motion of $M(p_1)$ eventually stays in $\bigcup_{\mathscr{A}(g) < \mathscr{B}(g)} \tilde{\tilde{S}}(g)$ whereas Brownian motion on $M(p_2)$ eventually stays in $\bigcup_{\mathscr{A}(g) > \mathscr{B}(g)} \tilde{\tilde{S}}(g)$ and these two sets are disjoint.*

Proof. As before, we first operate on L. Let p_1 be small so that the arguments of §3 prove that the translation invariant walk on Γ given by $\mathbb{P}_{p_1}$ finds the set $\{g \mid \mathscr{B}(g) > 2\mathscr{A}(g)\}$ absorbing. Now choose R to be large enough so that in the sense of §5 the Brownian motion on L gives rise to a nearly Markov walk on Γ to which the arguments of §3 also apply and enable one to deduce that $\bigcup_{\mathscr{A}(g) < 2\mathscr{B}(g)/3} \tilde{S}(g)$ is absorbing in $L(p_1)$. Repeat the procedure for p_2 chosen close to one with the role of a and b reversed.

Now we must compare the motions on M, L. Let z_t denote the Brownian motion on $M(p)$ which for definiteness has $z_0 \in \tilde{\tilde{Q}}(e)$. Then projecting to K and taking the unique lift in L which starts in $\tilde{Q}(e)$ we obtain a Brownian motion ${}^L z_t$ on L started in $\tilde{Q}(e)$. Of course z_t, ${}^L z_t$ are far from independent—they are coupled in a rather precise way. Suppose $z_s \in \tilde{\tilde{Q}}(g)$ and the first $\tilde{\tilde{Q}}(h)$ ($|h| \neq |g|$) visited by z after time s is $\tilde{\tilde{Q}}(g')$. Then $|g'| - |g| \in \{-1, 1\}$. Moreover if R is chosen large enough we can obtain

$$\mathbb{P}(|g'| > |g|) > (1 - \varepsilon) \min\left(\frac{2p}{1-p}, \frac{2(1-p)}{p}\right) \mathbb{P}(|g'| < |g|)$$

independent of the values of z_t, g, etc. Choose ε so that

$$(1 - \varepsilon) \min\left(\frac{2p_i}{1-p_i}, \frac{2(1-p_i)}{p_i}\right) > 1 + \delta, \qquad i = 1, 2.$$

Let s_1 be the first time z_t leaves $\tilde{\tilde{S}}(e)$, and let X_1 be the unique element g of Γ such that $z_{s_1} \in \tilde{\tilde{Q}}(g) \subset \tilde{\tilde{S}}(g)$. Let s_n be the first time z_t leaves $\tilde{\tilde{S}}(X_{n-1})$ after time s_{n-1}, and X_n the unique element h of Γ such that $z_{s_n} \in \tilde{\tilde{Q}}(h)$, etc. Combining the remarks of the last paragraph with an independent lower bound for the probability that $|X_{n+1}| \neq |X_n|$ we see that $\underline{\lim}_{n\to\infty}(|X_n|/n) > 0$. Following simpler but similar arguments to those in §3 we also see that

$$\underline{\lim}_{n\to\infty} (\mathscr{A}(X_n)/n) > 0 \quad \text{and} \quad \underline{\lim}_{n\to\infty} (\mathscr{B}(X_n)/n) > 0.$$

Suppose now that $(R_n, S_n)_{n=1}^{\infty}$ satisfy $n = o(R_n)$;[2] then two properties follow. First, there is a time $T < \infty$ after which there does not exist a triple k, $m < m'$ with $T < m < m'$, $|X_m| \in [R_k, S_k]$, and $|X_{m'}| \leqslant k$. Second, $\max\{k \mid \exists\, m < n,\ |X_m| \in [R_k, S_k]\}$ is $o(|X_n|)$ and $o(|n|)$.

Let ${}^L X_n$ be the unique $g \in \Gamma$ such that ${}^L z_{s_n} \in \tilde{Q}(g)$. Then looking back at the purely algebraic Lemma 6.2 we see that $|{}^L X_n^{-1} \cdot X_n| = o(|X_n|)$. It follows that

$$|\mathscr{A}({}^L X_n) - \mathscr{A}(X_n)| = o(|X_n|) = o(n) = o(\mathscr{A}(X_n)),$$
$$|\mathscr{B}({}^L X_n) - \mathscr{B}(X_n)| = o(\mathscr{B}(X_n)).$$

This proves the lemma.

We now wish to choose our sequences $[R_n, S_n]_{n=1}^{\infty}$ so that $M(p_i)$ has no positive harmonic functions, $i = 1, 2$. But before doing this we need some general discussion of boundary Harnack principles. A boundary Harnack principle takes the following from [1]:

Let h_1, h_2 be positive harmonic functions on some open set $U \subset R^d$, let V be an open set intersecting ∂U, and let K be a compact subset of $\overline{U}$ lying strictly inside V. Suppose h_1, h_2 are both zero on $\partial U \cap V$. Then h_1/h_2 extends to be continuous on K and there are absolute constants $c_1 < 1 < c_2$ such that

$$c_1 < \frac{h_1}{h_2}(x) \Big/ \frac{h_1}{h_2}(y) < c_2 \quad \text{for all } x, y \in K.$$

Such a principle does not in general hold without some conditions being imposed on ∂U. We are interested in the special case where U is a cylinder (or annulus) and h_1, h_2 are both positive, harmonic on the cylinder U, and zero at one end $\partial_1 U$ of U. In this case it is elementary to prove that h_1/h_2 has a C^∞ extension to $\partial_1 U$, moreover its derivatives on $\partial_1 U$ are all controlled in their size by the value of h_1/h_2 at any fixed point in $\partial_1 U \cup U$.

Consider the action of θ_n on $\{g \in \Gamma \mid |g| \geqslant n\}$. This action extends to $E_n \subset M$ by taking $\tilde{\tilde{S}}(g)$ to $\tilde{\tilde{S}}(g\theta_n)$. Checking that this is compatible with the identifications which make up M is quite routine except when $|g| \in [R_k, S_k]$

[2] This is our weak hypothesis on the (R_n, S_n).

for some $k < n$. In this case the construction of M identifies $C_\theta(g)$ with $C_{\theta^{-1}}(g\theta_k)$—to be compatible with the above θ_n action it is thus necessary that $C_\theta(g\theta_n)$ is identified with $C_{\theta^{-1}}((g\theta_k)\theta_n)$; this will only happen if $\theta_k\theta_n = \theta_n\theta_k$. Again this follows because $\theta_n^3 = \theta_{n-1}$.

We now prove the essential technical result, the statement is essentially equivalent to that of Proposition 6.3, but the proof is not the same.

Let $T(r)$ denote the first time z_t hits D_r, $t > 0$.

Proposition 8.2. *For each $\alpha > 1$, n, R_n there is an S (which might depend on the parameter p) such that if $S_n \geqslant S$ and F is any subset of D_{S_n} we have for all $x \in D_{R_n}$*

$$\alpha^{-1} < \frac{\mathbb{P}^x\left(z_{T(S_n)} \in F \mid T(R_n) > T(S_n)\right)}{\mathbb{P}^x\left(z_{T(S_n)} \in F\theta_n^j \mid T(R_n) > T(S_n)\right)} < \alpha \quad \forall j.$$

Remark 1. In other words if Brownian motion is started on the inner boundary of the annular region A_{R_n,S_n} and is conditioned to remain in A_{R_n,S_n} until it hits the outer boundary D_{S_n}, then it is essentially as likely to leave through any translate $F\theta_n$ of F as it is through F itself.

Remark 2. It might not be clear what is meant by Brownian motion started on D_{R_n} conditioned to leave A_{R_n,S_n} through D_{S_n}. To make this precise we reinterpret the statement in terms of positive harmonic functions. If $x \in A_{R_n,S_n}$ the conditonal probability is easily interpreted:

Let h_F be the positive harmonic function on A_{R_n,S_n} obtained by solving the Dirichlet problem with value 1 on F and zero elsewhere on $\partial A_{R_n,S_n}$. The claim is that the conditional probability

$$\mathbb{P}^x\left(z_{T(S_n)} \in F \mid T(R_n) > T(S_n)\right)$$

is just $h_F(x)/h_{D_{S_n}}(x)$ for $x \in A_{R_n,S_n}$. As both of these positive harmonic functions h_F, $h_{D_{S_n}}$ are zero on D_{R_n} and near D_{R_n}, A_{R_n,S_n} just looks like a union of cylinders we may apply the boundary Harnack principle to see that $h_F/h_{D_{S_n}}$ extends continuously to D_{R_n}. The value of this ratio is then what we mean by the above conditional probability when $x \in D_{R_n}$.

To obtain the claimed identification of the conditional probability with the ratio of positive harmonic functions let T denote the first time z quits A_{R_n,S_n}. Then $T(R_n) > T(S_n)$ is precisely the statement $z_T \in D_{S_n}$, but if z_T is in D_{S_n} then $z_{T(S_n)}$ is in F if and only if z_T is in F; so we restate our conditional probability as

$$\mathbb{P}^x\left(z_T \in F \mid z_T \in D_{S_n}\right) = \frac{\mathbb{P}^x\left(z_T \in F \text{ and } z_T \in D_{S_n}\right)}{\mathbb{P}^x\left(z_T \in D_{S_n}\right)}.$$

However $F \subset D_{S_n}$ so this equals

$$\frac{\mathbb{P}^x(z_T \in F)}{\mathbb{P}^x(z_T \in D_{S_n})} = \frac{h_F(x)}{h_{D_{S_n}}(x)}$$

as required. Of course this is just the normal Doob h-transform procedure—the only slightly novel remark is that the existence of a boundary Harnack principle allows one to start the process at a point x on the boundary providing the boundary values of the conditioning h are zero in a neighborhood of x.

Proof. We wish to prove that if x is in D then providing S_n is large enough one has

$$\alpha^{-1} < \frac{h_F(x)}{h_{D_{S_n}}(x)} \cdot \frac{h_{D_{S_n}}(x)}{h_{F\theta_n^j}(x)} < \alpha.$$

Suppose it were not true; fix $\alpha > 0$ and let $S_n \to \infty$. A simple compactness argument (using the equi-continuity provided by the boundary and usual Harnack principles) shows the existence of $x \in D_{R_n}$, $j < 3^{n-1}$, and h positive harmonic on E_{R_n} (where S_n is taken to be ∞ in the construction of M) such that $h(x\theta_n^j)/h(x) > \alpha$. We prove this to be impossible.

The positive harmonic functions on E_{R_n} which are zero on D_{R_n} and are normalized to be one at some fixed point form a compact convex set (again this uses BHP). It is enough to prove that the extremal ones are invariant under θ_n—for then they all are. But if h is extremal this means it is minimal, so that if $0 \leqslant \tilde{h} < ch$, then $\tilde{h} = dh$ for some constants c, d.

In E_{R_n} with the hypothesis that $S_n = \infty$ we have a uniform estimate for $y \in E_{R_n}$ of $d(y, y\theta_n)$, and because we have uniform estimates on curvature. etc. this can be translated into a uniform Harnack estimate: for all $h > 0$ harmonic on E_{R_n} and zero on D_{R_n} we have $h(y) > Ch(y\theta_n)$ for all $y \in E_{R_n}$ and some fixed C. It follows that if h is minimal, then $h(y\theta_n^j) = \phi(j)h(y)$ $\forall y$, where ϕ is a real character on the cyclic group of order 3^{n-1}. Of course the only such character is the constant one so $h(\cdot\theta_n^j) = h(\cdot)$ as required. This completes the proof.

We may use Proposition 8.2 to define the $(R_n, S_n)_{n=1}^{\infty}$ inductively. Choose R_1 arbitrarily and fix $\alpha > 1$. Suppose R_n is determined; let $S(p_1)$, $S(p_2)$ be the values of S determined by 8.2 with this value of α and the two choices of p determined earlier. Choose $S_n = \max(S(p_1), S(p_2))$ and $R_{n+1} > S_n$ so that $n = o(R_n)$. We claim that $M(p_i)$, $i = 1,2$, as constructed will have no positive harmonic functions.

To prove that M admits no nonconstant positive harmonic functions we proceed from Proposition 8.2 as we did from Proposition 6.3 in §6 for the Markov chain case. We prove

Proposition 8.3. *There is a β such that for any choice of n, $x, y \in B_n$, and $F \subset D_{S_n}$ the estimate*

$$\frac{1}{\beta} < \frac{\mathbb{P}^x\left(z_{T(S_n)} \in F\right)}{\mathbb{P}^y\left(z_{T(S_n)} \in F\right)} < \beta$$

holds.

Remark. It follows from this proposition that any positive harmonic function h on B_{S_n} satisfies $1/\beta < h(x)/h(y) < \beta$ for all $x, y \in B_n$. But $\bigcup B_n = M$ and in the sense of [9] the pairs (B_n, B_{S_n}) form a cover by uniform Harnack pairs. It follows that any positive harmonic function defined on all of M is uniquely determined by its value at any single point $x \in \bigcap B_n$; this is non-empty and so the positive harmonic function is constant.

Proof. Although slightly technical, the following argument is in essence a repeat of the Markov chain argument. To simplify notation let T denote the first time z_t leaves B_{S_n}, τ_1 the last time before T that z_t leaves B_n, and τ_2 the last time before Q that z_t leaves B_{S_n}. Let $\tau_i = 0$ if z_t never enters B_n ($i = 1$) or B_{S_n} ($i = 2$). We may condition on the value of B_{τ_2} as follows. If $x \in A_{n,R^n}$, then

$$\mathbb{P}^x\left(z_Q \in F \mid \tau_1 = 0\right) = \mathbb{E}^x\left(\mathbb{P}^{z_\tau Q}\left(z_Q \in F \mid \tau_2 = 0\right) \mid \tau_1 = 0\right).$$

Using Lemma 8.2 we see that the choice of $[R_n, S_n]_{n=1}^{\infty}$ ensures that the integrand on the right-hand side varies by a factor at most α if F is replaced by $F\theta_n^j$ for any choice of j. It follows that we may extend the conclusion of 8.2 slightly to

$$\alpha^{-1} < \frac{\mathbb{P}^x\left(z_Q \in F \mid \tau_1 = 0\right)}{\mathbb{P}^x\left(z_Q \in F\theta_j^t \mid \tau_1 = 0\right)} < \alpha \quad \forall j \in \mathbb{Z},$$

for all x in A_{n,R_n}. But, because the θ action is defined outside B_n, we may move x rather than F and obtain

$$\alpha^{-1} < \frac{\mathbb{P}^x\left(z_Q \in F \mid \tau_1 > 0\right)}{P^{x\theta_n^j}\left(z_Q \in F \mid \tau_1 > 0\right)} < \alpha$$

for all $x \in A_{n,R_n}$ and all $j \leqslant 3^{j-1}$ (although we do not use it, the same estimates hold true of the unconditional probability that Z_t first exits A_{n,S_n} through F).

Let $O = \bigcup\{\tilde{\tilde{Q}}(g) \mid |g| = n\}$, and partition O into O_1, O_2, O_3, O_4 according to the leftmost letter of g in reduced form. Let H be the first time z_t hits O and $\hat{H}$ the first time that z_t hits O after τ_1 (i.e. after finally entering A_{n,s_n}). Suppose x is in B_n; by conditioning on the value of z_H we have

$$\mathbb{P}^x(z_Q \in F) = \mathbb{E}^x\left(\mathbb{P}^{z_{\hat{H}}}(z_Q \in F \mid \tau_1 = 0)\right).$$

But we may apply Harnack to see that the value of $\mathbb{P}^y(z_Q \in F \mid \tau_1 = 0)$ does not change by more than a bounded factor β as y varies over $\tilde{\tilde{Q}}(g)$ for some fixed g with $|g| = n$ (β is independent of g, n, etc.). The earlier remarks guarantee that as one changes x to $x\theta_n^j$ one does not change the value of the expression by more than a factor α. So we have that

$$(\alpha\beta)^{-1}\left(\sum_1^4 \mathbb{P}^x(z_{\hat{H}} \in O_i)\mathbb{P}^{x_i}(z_Q \in F \mid \tau_1 = 0)\right)$$
$$\leqslant \mathbb{P}^x(z_Q \in F) \leqslant (\alpha\beta)\left(\sum_1^4 \mathbb{P}^x(z_{\hat{H}} \in O_i)\mathbb{P}^{x_i}(z_Q \in F \mid \tau_1 = 0)\right),$$

where x_i is any fixed point in O_i.

To finish the argument we claim that there is a universal lower bound γ on $\mathbb{P}^x(z_{\hat{H}} \in O_i)$ for $x \in B_n$ and $i = 1 \cdots 4$. For then we have

$$(\alpha\beta)^{-1}\gamma\sum_1^4 \mathbb{P}^{x_i}(z_Q \in F \mid \tau_1 = 0)$$
$$\leqslant \mathbb{P}^x(z_Q \in F) \leqslant (\alpha\beta)\sum_1^4 \mathbb{P}^{x_i}(z_Q \in F \mid \tau_1 = 0)$$

for all $x \in B_n$; in other words to within a factor $(\alpha\beta)^2/\gamma$ the probability of the unconditional process hitting F does not depend on $x \in B_n$. This proves the theorem subject to our claimed lower bound on $\mathbb{P}^x(z_{\hat{H}} \in O_i)$. First we recall that when considering the discrete skeleton $X_i \in \Gamma$ of z_t, we saw that there is an independent lower bound on the probability of the length of X_i ever decreasing below its current value independent of the value of X_i. The same arguments also prove that (for large enough R) there is a δ such that if z_0 is in O, then the probability of z_t ever reentering the ball B_n (which is slightly separated from O) is at most $(1 - \delta < 1)$ independent of the value of z_0, n. It follows that with probability at least δ (where δ is independent of the value of z_H) we have $\hat{H} = H$. So $\delta\mathbb{P}^x(z_H \in O_i) < \mathbb{P}^x(z_{\hat{H}} \in O_i)$. It is enough to give a lower bound ε on the value of $\mathbb{P}^x(z_H \in O_i)$ independent of $x \in B_n$, i, n and then put $\gamma = \delta\varepsilon$.

Let $h^i(x) = \mathbb{P}^x(z_H \in O_i)$, h^i is positive and harmonic off O. D_n is a union of circles each of which bisects a cylinder terminating at O. If the cylinder terminates at O_i, then h has value at least $\frac{1}{2}$ on that circle. However we have a uniform upper estimate $(6 \cdot \mathrm{Diam}(K))$ on the distance within D_n from any one of the $4 \cdot 3^{n-1}$ boundary circles in D_n and the nearest of the special ones which bisect cylinders terminating in O_i (3^{n-1} in all). We also have a uniform lower bound of $R/2$ on the distance from the boundary of B_n to the edge of the domain of harmonicity of h. It follows from the constant negative curvature of M (lift to the universal cover) that these distance estimates can be turned into Harnack estimates; using these we obtain a lower estimate ε for the value of h^i on D_n independent of n, i. But by the minimum principle this estimate extends to B_n and we have the estimates

$$h^i(x) > \varepsilon \quad \forall x \in B_n, \qquad \mathbb{P}^x(z_{\hat{H}} \in O_i) \geqslant \delta\varepsilon = \mu,$$

as we required.

This proves that $M(p_i)$ has no nonconstant positive harmonic functions. It is probable that the reader will be put off by the technicality of the above argument. He should not be—the main point is that once the Brownian traveller has emerged from B_n for the last time the only thing which really affects his probability of hitting $F \subset D_{S_n}$ is which of the four equivalence classes of words of length n he chooses to enter A_{n,S_n} through. Because there are only four and they are all evenly spread out through B_n the probability of last exiting through O_i is much the same as the probability of last exiting through O_j, $j \neq i$. Unfortunately one has to prove it.

It is plain sailing to prove that $N(p_1)$ admits precisely two linearly independent positive harmonic functions (both bounded of course) and $N(p_2)$ admits only the constant functions. One simply repeats the arguments of §7 with small changes. First reflection is replaced by $\mathscr{P}$ action (so $h^r(x) = h(\mathscr{P}x)$, etc.). Then the argument for $i = 1$ goes through word-for-word; for $i = 2$ one follows §7 to prove that any minimal harmonic function is bounded. Then one uses Harnack to prove that if $h + h^r = 2c$, then we have $c/\theta < h(z) < c\theta$ for all $z \in \bigcup_{x \in \Gamma \times C_2, \mathscr{A}(x) > \mathscr{B}(x)} S(x)$, but the latter is an absorbing set in $M(p_2)$ so the estimate propogates to all $z \in M(p_2)$. In particular h cannot be minimal unless it is constant (because it is subordinated by c/θ). This argument would also work in §7 but the appeal to Harnack is not necessary there.

Acknowledgment. The author is very grateful for the helpful discussions he has had with many people while preparing this—particularly B. Baumslag, D. Drasin, and O. Pretzel. After the main part of this manuscript was prepared and while the author was at a probability problem session in Swansea, he

raised the question of what happens to $\lim_{n\to\infty}(\mathscr{A}(X_n)/\mathscr{B}(X_n))$ when p is close to $\frac{1}{2}$ for the simple random walk case. This was solved completely in the ensuing discussion mainly by M. Barlow, T. K. Carne, W. Kendall and D. Williams. It will be published as a separate note.

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